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IV.

On a Short Process for Solving the Irreducible Case of Cardan's Method.

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The equation having three commensurable roots, a, b, c , is

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0.$$

Reducing the roots of this equation by $\frac{1}{3}(a + b + c)$, we have

$$y^3 - \frac{1}{3}(a^2 + b^2 + c^2 - ab - ac - bc)y - \frac{1}{27}(2a^3 + 2b^3 + 2c^3 - 3a^2b - 3a^2c - 3ab^2 - 3ac^2 - 3b^2c - 3bc^2 + 12abc) = 0.$$

This being of the form $y^3 + my + n = 0$, we have, substituting in Cardan's formula and reducing,

$$y = \frac{1}{3} \sqrt[3]{-\frac{27}{2}n + \frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}} + \frac{1}{3} \sqrt[3]{-\frac{27}{2}n - \frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}} = u + v.$$

Since u is a binomial imaginary, its cube root will be of the form $\alpha + \sqrt{-\beta}$, and $\sqrt{\frac{\beta}{3}}$ will be rational. Hence

$$(1) \quad \alpha(\alpha^2 - 3\beta) = \frac{1}{2}(2a^3 + 2b^3 + 2c^3 - 3a^2b - 3a^2c - 3ab^2 - 3ac^2 - 3b^2c - 3bc^2 + 12abc),$$

$$(2) \quad \sqrt{-\beta}(3\alpha^2 - \beta) = \frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3} = (r).$$

Since α must be rational, $\sqrt{-\beta}$ must be of the first degree with reference to a, b, c , and the only factors of (r) of that degree are of the form $\frac{(a-b)\sqrt{-3}}{p}$, where p is some integer: p must be 2, for substituting $\sqrt{-\beta} = \frac{b-c}{p}\sqrt{-3}$

in (2) and reducing, we have

$$\alpha^2 = \frac{p}{2}(a^2 - ab - ac + bc) + \frac{b^2 - 2bc + c^2}{p^2},$$

which will not give a rational value to α unless $p = 2$.

Let us assume then $\sqrt{-\beta} = \frac{b-c}{2}\sqrt{-3}$, substitute in (2) and reduce.

We find $\alpha = \pm \left(a - \frac{b+c}{2}\right)$, and by substitution in (1), $\alpha = a - \frac{b+c}{2}$, and similarly for the other factors of (r) . Hence

$$u = \frac{1}{3} \left(a - \frac{b+c}{2} + \frac{b-c}{2} \sqrt{-3} \right), \frac{1}{3} \left(b - \frac{a+c}{2} + \frac{a-c}{2} \sqrt{-3} \right), \text{ and } \frac{1}{3} \left(c - \frac{a+b}{2} + \frac{a-b}{2} \sqrt{-3} \right); \text{ similarly}$$

$$v = \frac{1}{3} \left(a - \frac{b+c}{2} - \frac{b-c}{2} \sqrt{-3} \right), \frac{1}{3} \left(b - \frac{a+c}{2} - \frac{a-c}{2} \sqrt{-3} \right), \text{ and } \frac{1}{3} \left(c - \frac{a+b}{2} - \frac{a-b}{2} \sqrt{-3} \right);$$

$$y = \frac{1}{3} (2a - b - c), \frac{1}{3} (2b - a - c), \text{ and } \frac{1}{3} (2c - a - b); x = a, b, \text{ and } c.$$

The three values of u are connected, as they should be, by the ratios

$$1 : \frac{1}{2} (-1 + \sqrt{-3}) : \frac{1}{2} (-1 - \sqrt{-3}).$$

If two of the roots of the equation are equal, as b and c , then $(r) = 0$ and $-\frac{1}{2}n = (a-b)^3$; and if two are imaginary, as b and c , $(= \gamma \pm \delta \sqrt{-3})$, (r) becomes rational and $-\frac{1}{2}n + (r) = (a - \gamma - 3\delta)^3$, showing why, in these two cases, Cardan's formula gives a rational result.

Numerical Equations.

$$1. \quad x^3 - 9x^2 + 14x + 24 = 0.$$

Reduce the roots by 3; then $y^3 - 13y + 12 = 0$, and by Cardan's formula,

$$y = \sqrt[3]{-6 + \frac{35}{3}\sqrt{-\frac{1}{3}}} + \sqrt[3]{-6 - \frac{35}{3}\sqrt{-\frac{1}{3}}};$$

$$(1) \quad \alpha(\alpha^2 - 3\beta) = -6, \quad (2) \quad \sqrt{-\beta}(3\alpha^2 - \beta) = \frac{35}{3}\sqrt{-\frac{1}{3}};$$

$\sqrt{-\beta} = \sqrt{-\frac{1}{3}}, \frac{5}{2}\sqrt{-\frac{1}{3}}, \text{ and } -\frac{7}{2}\sqrt{-\frac{1}{3}}; \beta = \frac{1}{3}, \frac{25}{12}, \text{ and } \frac{49}{12};$ then by (2), $\alpha = \pm 2, \pm \frac{3}{2}, \text{ and } \pm \frac{1}{2};$ and by (1), $\alpha = -2, \frac{3}{2}, \text{ and } \frac{1}{2}.$

$\therefore y = -2 + \sqrt{-\frac{1}{3}} + \left(-2 - \sqrt{-\frac{1}{3}}\right) = -4, y = \frac{3}{2} + \frac{5}{2}\sqrt{-\frac{1}{3}} + \frac{3}{2} - \frac{5}{2}\sqrt{-\frac{1}{3}} = 3, \text{ and } y = \frac{1}{2} - \frac{7}{2}\sqrt{-\frac{1}{3}} + \frac{1}{2} + \frac{7}{2}\sqrt{-\frac{1}{3}} = 1; x = -1, 6, \text{ and } 4.$

$$2. \quad x^3 - 16x^2 + 73x - 90 = 0.$$

Reduce the roots by $\frac{16}{3}$; then $y^3 - \frac{37}{3}y - \frac{110}{27} = 0$, and by Cardan's formula,

$$y = \frac{1}{3} \sqrt[3]{55 + 126\sqrt{-3}} + \frac{1}{3} \sqrt[3]{55 - 126\sqrt{-3}}.$$

Since $\frac{126}{27} \sqrt{-3} = \frac{3}{2} (a-b)(b-c)(a-c) \sqrt{-3}$, the factors are probably

$\pm 2 \sqrt{-3}$, $\pm \frac{3}{2} \sqrt{-3}$ and $\pm \frac{7}{2} \sqrt{-3}$. Hence

$$(1) \quad \alpha (\alpha^2 - 3\beta) = 55, \quad (2) \quad \sqrt{-\beta} (3\alpha^2 - \beta) = 126 \sqrt{-3};$$

$$\sqrt{-\beta} = 2 \sqrt{-3}; \quad 3\alpha^2 = 75, \quad \alpha = \pm 5; \quad \alpha = -5;$$

$$\sqrt{-\beta} = \frac{3}{2} \sqrt{-3}; \quad 3\alpha^2 = \frac{363}{4}, \quad \alpha = \pm \frac{11}{2}; \quad \alpha = \frac{11}{2};$$

$$\sqrt{-\beta} = -\frac{7}{2} \sqrt{-3}; \quad 3\alpha^2 = \frac{3}{4}, \quad \alpha = \pm \frac{1}{2}; \quad \alpha = -\frac{1}{2};$$

$$y = \frac{1}{3} (-5 + 2\sqrt{-3}) + \frac{1}{3} (-5 - 2\sqrt{-3}) = -\frac{10}{3},$$

$$y = \frac{1}{3} \left(\frac{11}{2} + \frac{3}{2} \sqrt{-3} \right) + \frac{1}{3} \left(\frac{11}{2} - \frac{3}{2} \sqrt{-3} \right) = \frac{11}{3},$$

$$y = \frac{1}{3} \left(-\frac{1}{2} - \frac{7}{2} \sqrt{-3} \right) + \frac{1}{3} \left(-\frac{1}{2} + \frac{7}{2} \sqrt{-3} \right) = -\frac{1}{3};$$

$x = 2, 9, \text{ and } 5.$

V.

An Extension of Taylor's Theorem

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$$f(x+a) - f(x) = \int_0^a da \frac{d}{dx} f(x+a), \quad \left(1 - \int_0^a da \frac{d}{dx}\right) f(x+a) = f(x),$$

$$\therefore f(x+a) = \left(1 - \int_0^a da \frac{d}{dx}\right)^{-1} f(x). \quad \text{I.}$$

$$\left(1 - \int_0^a da \frac{d}{dx}\right)^{-1} \int_0^a da f(x+b) = \int_0^a da \left(1 - \int_0^a da \frac{d}{dx}\right)^{-1} f(x+b)$$

$$(\text{by I.}) \quad = \int_0^a da \left\{ 1 - \int_0^{a+b} d(a+b) \frac{d}{dx} \right\}^{-1} f(x). \quad \text{II.}$$

Expanding by I, but modifying each remainder by II before proceeding to obtain the term next following, we get

$$\begin{aligned} f(x+a+b+c+e+\&c.) &= f(x+b+c+e+\&c.) + \int_0^a da f'(x+c+e+\&c.) \\ &+ \int_0^a da \int_0^{a+b} d(a+b) f''(x+e+\&c.) \\ &+ \int_0^a da \int_0^{a+b} d(a+b) \int_0^{a+b+c} d(a+b+c) f'''(x+\&c.) + \&c., \end{aligned} \quad \text{III.}$$

which is the extension proposed.